Jones-Makarov's "On density properties of harmonic measure" revisited

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Abstract. We study the boundary properties of conformal mappings, following the trail blazed by Jones and Makarov. It is our intention to tie up their method with the traditional approach in conformal mapping. Also, we extend the Jones-Makarov theorem on the local behavior of universal integral means spectrum near point 2 to complex values of the parameter.

1. Introduction

Integral means spectra. The class of univalent functions (conformal mappings) $\varphi : \mathbb{D} \to \mathbb{C}$, subject to the normalizations $\varphi(0) = 0$ and $\varphi'(0) = 1$, is denoted by $\mathcal{S}$. Its subclass consisting of bounded functions is denoted $\mathcal{S}_b$.

Recall that, for $t \in \mathbb{R}$, the integral means $M_t[\varphi'](r)$ are defined by

$$M_t[\varphi'](r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi'(re^{i\theta})|^t \, \mathrm{d}\theta, \qquad 0 < r < 1.$$

This definition can be extended to complex parameters if we put

$$M_\tau[\varphi'](r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \left[\varphi'(re^{i\theta}) \right]^\tau \right| \mathrm{d}\theta, \qquad 0 < r < 1,$$

when $\tau \in \mathbb{C}$. Now, for a fixed τ, we define the real number $\beta_\varphi(\tau)$ as the infimum of positive β such that

$$(1.1) \qquad M_\tau[\varphi'](r) = O\left(\frac{1}{(1-r)^\beta} \right), \qquad r \to 1^-;$$

we note that (1.1) holds for sufficiently large positive β. The function $\beta_\varphi(\tau)$ will be referred to as *the integral means spectrum* of φ. For real t the quantity $\beta_\varphi(t)$ measures the expansion or compression of the real disc near the boundary under the mapping φ. Considering a complex parameter τ, we also take into account the rotation. Finally, we define the *universal integral means spectra* for the classes $\mathcal{S}$ and $\mathcal{S}_b$ by

$$\mathrm{B}(\tau) = \sup_{\varphi \in \mathcal{S}} \beta_\varphi(\tau), \qquad \mathrm{B}_b(\tau) = \sup_{\varphi \in \mathcal{S}_b} \beta_\varphi(\tau).$$

For $-1 < \alpha < +\infty$, we introduce Bergman spaces $\mathcal{H}_\alpha^2(\mathbb{D})$ consisting of those holomorphic in $\mathbb{D}$ functions f for which

$$\|f\|_{\mathcal{H}_\alpha^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \, \mathrm{d}A(z) < +\infty,$$

where $\mathrm{d}A(z) = \pi^{-1}\mathrm{d}x\,\mathrm{d}y$, $z = x + iy$, is normalized two-dimensional Lebesgue measure. We point out the following almost obvious observation:

$$\beta_\varphi(\tau) = \inf\left\{\alpha + 1 : (\varphi')^{\tau/2} \in \mathcal{H}_\alpha^2(\mathbb{D})\right\}, \qquad \tau \in \mathbb{C}.$$

It is a long-standing problem to determine universal spectra $\mathrm{B}(\tau)$ and $\mathrm{B}_b(\tau)$. A solution is known only for special ranges of τ (see [3, 5]). The well-known Brennan conjecture states that $\mathrm{B}(-2) = 1$. For the recent progress on this problem, see [6], where the best known estimates for $\mathrm{B}(\tau)$ are found.

We mention also the following connection between the spectra B and B_b found by Nikolai Makarov [10]:

$$\mathrm{B}(t) = \max\left\{\mathrm{B}_b(t),\, 3t - 1\right\}, \qquad t \in \mathbb{R}.$$

In an unpublished manuscript [2] I. Binder extends this formula to general complex τ:

$$\mathrm{B}(\tau) = \begin{cases} \mathrm{B}_b(\tau), & \operatorname{Re}\tau \le 0, \\ \max\left\{\mathrm{B}_b(\tau), |\tau| + 2\operatorname{Re}\tau - 1)\right\}, & \operatorname{Re}\tau > 0. \end{cases}$$

The paper of Jones and Makarov. One of the deepest results concerning integral means spectra is a theorem due to Peter Jones and Nikolai Makarov [8] regarding the local behavior of B_b near the point 2. Note that for $\varphi \in \mathcal{S}$ we have $\varphi \in \mathcal{S}_b$ if and only if $\varphi' \in \mathcal{H}_0^2(\mathbb{D})$. It is proved in [8] that for $\varphi \in \mathcal{S}_b$, there is an absolute constant $C_0 > 0$ such that for sufficiently small positive ε and any $C > C_0$,

$$(1.2) \qquad \int_{\mathbb{D}} |\varphi'(z)|^{2-\varepsilon}(1 - |z|^2)^{-\varepsilon+C\varepsilon^2} \, \mathrm{d}A(z) < +\infty.$$

The order is sharp in the sense that (1.2) fails for some positive C. In terms of the integral means spectrum, (1.2) means that $\mathrm{B}_b(2 - \varepsilon) = 1 - \varepsilon + O(\varepsilon^2)$, $\varepsilon \to 0^+$.

The (very difficult) proof of (1.2) is based on subtle properties of harmonic measure.

Results. In this note we try to connect this result with the more classical methods of geometric univalent function theory. As an application of our approach we extend the Jones-Makarov result to complex values. Namely, we prove the following theorem:

Theorem 1.1. *There is an absolute constant $C > 0$ such that for $\varphi \in \mathcal{S}_b$ and for $\tau \in \mathbb{C}$ with sufficiently small $|\tau|$, we have*

$$(1.3) \qquad \int_{\mathbb{D}} \left|\left(z\frac{\varphi'(z)}{\varphi(z)}\right)^{2-\tau}\right| (1 - |z|^2)^{-\operatorname{Re}\tau + C|\tau|^2 \log\frac{1}{|\tau|}} \, \mathrm{d}A(z) < +\infty.$$

As a corollary to (1.3) we obtain the following statement on the local behavior of B_b:

Corollary 1.2. *We have*

$$(1.4) \qquad \mathrm{B}_b(2 - \tau) = 1 - \operatorname{Re}\tau + O\left(|\tau|^2 \log\frac{1}{|\tau|}\right), \qquad \tau \in \mathbb{C},\ |\tau| \to 0^+.$$

Estimate (1.4) shows that we have essentially smaller growth of B_b along the imaginary axis, than along $\mathbb{R}$. Note that for small $\tau > 0$ (1.4) is only slightly weaker than Jones-Makarov's estimate (up to a logarithmic factor in the error term).

We mention also that (1.4) is consistent with the so-called Kraetzer-Binder conjecture [9].

Discussion of methods.

2. Transferred Cauchy and Beurling transforms

The Cauchy transform and the Beurling transform. Let Ω be a bounded domain in the complex plane $\mathbb{C}$. For Lebesgue area integrable functions f on Ω, we define the following two integral operators, the *Cauchy transform* $\mathfrak{C}_\Omega$,

$$\mathfrak{C}_\Omega[f](z) = \int_\Omega \frac{f(w)}{w - z}\, \mathrm{d}A(w),$$

and the *conjugate Cauchy transform* $\bar{\mathfrak{C}}_\Omega$

$$\bar{\mathfrak{C}}_\Omega[f](z) = \int_\Omega \frac{f(w)}{\bar{w} - \bar{z}}\, \mathrm{d}A(w).$$

It is clear that in the sense of distribution theory,

$$\bar{\partial}_z \mathfrak{C}_\Omega[f](z) = -f(z), \qquad z \in \Omega,$$

and

$$\partial_z \bar{\mathfrak{C}}_\Omega[f](z) = -f(z), \qquad z \in \Omega.$$

Associated to the Cauchy transform is the *Beurling transform*

$$\mathfrak{B}_\Omega[f](z) = \partial_z \mathfrak{C}_\Omega[f](z) = \mathrm{pv} \int_\Omega \frac{f(w)}{(w - z)^2}\, \mathrm{d}A(w), \qquad z \in \Omega,$$

while to the conjugate Cauchy transform we associate the *conjugate Beurling transform*

$$\bar{\mathfrak{B}}_\Omega[f](z) = \bar{\partial}_z \bar{\mathfrak{C}}_\Omega[f](z) = \mathrm{pv} \int_\Omega \frac{f(w)}{(\bar{w} - \bar{z})^2}\, \mathrm{d}A(w), \qquad z \in \Omega.$$

It is well-known that for $\Omega = \mathbb{C}$, both $\mathfrak{B}_\mathbb{C}$ and $\bar{\mathfrak{B}}_\mathbb{C}$ are unitary transformations $L^2(\mathbb{C}) \to L^2(\mathbb{C})$. Moreover, the adjoint $\mathfrak{B}_\mathbb{C}^*$ of $\mathfrak{B}_\mathbb{C}$ coincides with $\bar{\mathfrak{B}}_\mathbb{C}$, so that $\bar{\mathfrak{B}}_\mathbb{C}\mathfrak{B}_\mathbb{C}$ and $\mathfrak{B}_\mathbb{C}\bar{\mathfrak{B}}_\mathbb{C}$ equal the identity operator on $L^2(\mathbb{C})$. These assertions remain valid in case $\mathbb{C} \setminus \Omega$ has zero area. For general Ω, however, $\mathfrak{B}_\Omega$ and $\bar{\mathfrak{B}}_\Omega$ are contractions $L^2(\Omega) \to L^2(\Omega)$, with $\mathfrak{B}_\Omega^* = \bar{\mathfrak{B}}_\Omega$.

As for the adjoint of $\mathfrak{C}_\Omega$ with respect to the standard $L^2(\Omega)$-inner product

$$\langle f, g \rangle_\Omega = \int_\Omega f(z)\, \bar{g}(z)\, \mathrm{d}A(z),$$

we readily compute that $\mathfrak{C}_\Omega^* = -\bar{\mathfrak{C}}_\Omega$.

Let $W^{1,2}(\Omega)$ be the space of all functions $f \in L^2(\Omega)$ such that $\partial f \in L^2(\Omega)$ and $\bar{\partial} f \in L^2(\Omega)$ (the derivatives are understood in distributional sense). We denote by $W^{1,2}(\Omega)/\mathbb{C}$ the Hilbert-Dirichlet space (modulo the constant functions) of functions f on Ω with norm (the Dirichlet integral)

$$\|f\|_{W^{1,2}(\Omega)/\mathbb{C}}^2 = \frac{1}{2} \int_\Omega \{|\partial f(z)|^2 + |\bar{\partial} f(z)|^2\} \mathrm{d}A(z).$$

Then $\mathfrak{C}_\Omega$ and $\bar{\mathfrak{C}}_\Omega$ are contractions $L^2(\Omega) \to W^{1,2}(\Omega)/\mathbb{C}$; in fact, this is an equivalent way of saying that $\mathfrak{B}_\Omega$ and $\bar{\mathfrak{B}}_\Omega$ are contractions $L^2(\Omega) \to L^2(\Omega)$.

Proposition 2.1. *For $f \in L^2(\Omega)$, extended to vanish on $\mathbb{C} \setminus \Omega$, we have $\mathfrak{C}_{\mathbb{C}}[f] \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$.*

By the Sobolev inequality, functions in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ belong to all $L^q_{\mathrm{loc}}(\mathbb{C})$, $1 < q < +\infty$. This may be improved substantially (see Chapter 3 of the book [1]).

Proposition 2.2. *For each $g \in W^{1,2}(\mathbb{C})$ of norm ≤ 1, we have $\exp[\beta_0 |g|^2] \in L^1_{\mathrm{loc}}(\mathbb{C})$, for some positive absolute constant β_0.*

Conformal mapping and transferred Cauchy transforms. If Ω is simply connected (not the whole plane), there exists a conformal mapping $\varphi : \mathbb{D} \to \Omega$ that is onto. We connect two functions f and g, on Ω and $\mathbb{D}$, respectively, via

$$g(z) = \bar{\varphi}'(z) \, f \circ \varphi(z),$$

and define the integral operator

$$\mathfrak{C}_{\varphi}[g](z) = (\mathfrak{C}_{\Omega}[f]) \circ \varphi(z) = \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \, g(w) \, \mathrm{d}A(w), \qquad z \in \mathbb{D};$$

$\mathfrak{C}_{\varphi}$ is then a contraction $L^2(\mathbb{D}) \to W^{1,2}(\mathbb{D})$, as follows from the previous observation that $\mathfrak{C}_{\Omega}$ is a contraction $L^2(\Omega) \to W^{1,2}(\Omega)$. Analogously, we define

$$\bar{\mathfrak{C}}_{\varphi}[g](z) = \int_{\mathbb{D}} \frac{\bar{\varphi}'(w)}{\bar{\varphi}(w) - \bar{\varphi}(z)} \, g(w) \, \mathrm{d}A(w), \qquad z \in \mathbb{D},$$

and realize that $\bar{\mathfrak{C}}_{\varphi}$ is a contraction $L^2(\mathbb{D}) \to W^{1,2}(\mathbb{D})$.

The transferred Beurling transforms. It is well-known that $\mathfrak{B}_{\mathbb{C}}$ acts boundedly on $L^p(\mathbb{C})$, for all p with $1 < p < +\infty$. Let $K(p)$ be a positive constant such that

$$(2.1) \qquad\qquad \|\mathfrak{B}_{\mathbb{C}} f\|_{L^p(\mathbb{C})} \leq K(p) \, \|f\|_{L^p(\mathbb{C})}, \qquad f \in L^p(\mathbb{C});$$

for instance, we could use the operator norm to get the optimal constant $K(p)$. For $0 \leq \theta \leq 2$, we introduce the *θ-skewed Beurling transform*, as defined by

$$\mathfrak{B}^{\theta}_{\varphi}[f] = \mathrm{pv} \int_{\mathbb{D}} \frac{\varphi'(z)^{\theta} \varphi'(w)^{2-\theta}}{(\varphi(z) - \varphi(w))^2} \, f(w) \, \mathrm{d}A(w).$$

It follows from (2.1) that

$$\left\| \mathfrak{B}^{2/p}_{\varphi} f \right\|_{L^p(\mathbb{D})} \leq K(p) \, \|f\|_{L^p(\mathbb{D})}, \qquad f \in L^p(\mathbb{D}),$$

for all p with $1 < p < +\infty$. In the symmetric case $\theta = 1$, we shall write $\mathfrak{B}_{\varphi}$ in place of $\mathfrak{B}^1_{\varphi}$. We note that $\mathfrak{B}_{\varphi}$ is a contraction on $L^2(\mathbb{D})$.

3. An operator identity related to the Grunsky inequality

The basic identity. In case $\varphi(z) = z$, we write $\mathfrak{C}$ instead of $\mathfrak{C}_{\varphi}$, and $\bar{\mathfrak{C}}$ in place of $\bar{\mathfrak{C}}_{\varphi}$. Likewise, under the same circumstances, we write $\mathfrak{B}$ and $\bar{\mathfrak{B}}$ instead of $\mathfrak{B}_{\varphi}$ and $\bar{\mathfrak{B}}_{\varphi}$. Next, let $\mathfrak{P} : L^2(\mathbb{D}) \to A^2(\mathbb{D})$ be the orthogonal projection,

$$\mathfrak{P}f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} \, \mathrm{d}A(w),$$

and let $\mathfrak{I}_0$ be the operation on analytic functions in $\mathbb{D}$ defined by

$$\mathfrak{I}_0[f](z) = \int_0^z f(w) \, \mathrm{d}w, \qquad z \in \mathbb{D}.$$

Analogously, we let $\bar{\mathfrak{P}}$ be the orthogonal projection to the antianalytic functions in $L^2(\mathbb{D})$, and we let $\bar{\mathfrak{J}}_0$ be the corresponding integration operator acting on the antianalytic functions. In terms of formulas, we have

$$\bar{\mathfrak{P}}f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - w\bar{z})^2} \, dA(w),$$

and

$$\bar{\mathfrak{J}}_0[f](z) = \int_0^z f(w) \, d\bar{w}, \qquad z \in \mathbb{D}.$$

The following identity is basic to our investigation.

Proposition 3.1. *For $\varphi \in \mathcal{S}$, we have the identity*

$$(3.1) \qquad \log \frac{z(\varphi(z) - \varphi(\zeta))}{(z - \zeta)\varphi(z)} + \log(1 - \bar{z}\zeta) = \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\zeta}{1 - \bar{w}\zeta} \, dA(w).$$

Proof. We note that for analytic functions f area-integrable in $\mathbb{D}$, we have

$$\int_{\mathbb{D}} f(w) \frac{\zeta}{1 - \bar{w}\zeta} \, dA(w) = \int_0^\zeta f(w) \, dw.$$

It follows that

$$\int_{\mathbb{D}} \left[\frac{\varphi'(w)}{\varphi(w) - \varphi(z)} - \frac{1}{w - z} \right] \frac{\zeta}{1 - \bar{w}\zeta} \, dA(w) = \log \frac{z(\varphi(z) - \varphi(\zeta))}{(z - \zeta)\varphi(z)}.$$

Next, we compute that

$$\int_{\mathbb{D}} \frac{\zeta}{(w - z)(1 - \bar{w}\zeta)} \, dA(w) = \log\left(1 - \bar{z}\zeta\right).$$

The assertion is now immediate. $\qquad\square$

If we apply the differentiation operator ∂_ζ to the identity of Proposition 3.1, we get

$$\frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z} - \frac{\bar{z}}{1 - \bar{z}\zeta} = \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{1}{(1 - \bar{w}\zeta)^2} \, dA(w),$$

which in terms of operators may be written in the form

$$(3.2) \qquad \mathfrak{C}_\varphi - \mathfrak{C} - \bar{I}_0\bar{P} = \mathfrak{C}_\varphi\bar{P}.$$

As we apply the operator ∂ (differentiation with respect to z) to both sides, we obtain the derived identity

$$(3.3) \qquad \mathfrak{B}_\varphi - \mathfrak{B} = \mathfrak{B}_\varphi\bar{P}.$$

We claim that the identity (3.3) expresses in the strong Grunsky inequality in a rather detailed form. The main observation needed is that both $\mathfrak{B}_\varphi$ and $\bar{P}$ are contractions on $L^2(\mathbb{D})$, making their product $\mathfrak{B}_\varphi\bar{P}$ a contraction as well. If f is an antianalytic function in $L^2(\mathbb{D})$, then

$$(3.4) \qquad \left\|(\mathfrak{B}_\varphi - \mathfrak{B})f\right\|_{L^2(\mathbb{D})}^2 = \left\|\mathfrak{B}_\varphi\bar{P}f\right\|_{L^2(\mathbb{D})}^2 \leq \|f\|_{L^2(\mathbb{D})}^2.$$

It is an easy exercise to check that (3.4) is equivalent to the strong Grunsky inequality as formulated in [4].

For the applications we have in mind, it will be convenient to work with the following variant of Proposition 3.1.

Proposition 3.2. *We have the identity*

$$\log \frac{z(\varphi(z) - \varphi(\zeta))}{(z - \zeta)\varphi(z)} - \zeta(1 - |\zeta|^2)\left[\frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z}\right] + \log\left(1 - \bar{z}\zeta\right) + \bar{z}\zeta$$

$$= \zeta^2 \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\bar{\zeta} - \bar{w}}{(1 - \bar{w}\zeta)^2}\, \mathrm{d}A(w).$$

Proof. We have that

$$\zeta^2 \frac{\bar{\zeta} - \bar{w}}{(1 - \bar{w}\zeta)^2} = \frac{\zeta}{1 - \bar{w}\zeta} - \frac{\zeta(1 - |\zeta|^2)}{(1 - \bar{w}\zeta)^2},$$

so that for analytic functions f area-integrable in $\mathbb{D}$, we get

$$\zeta^2 \int_{\mathbb{D}} f(w) \frac{\bar{\zeta} - \bar{w}}{(1 - \bar{w}\zeta)^2}\, \mathrm{d}A(w) = \int_0^\zeta f(w)\, \mathrm{d}w - \zeta(1 - |\zeta|^2)f(\zeta).$$

In particular, this leads to the identity

$$\zeta^2 \int_{\mathbb{D}} \left[\frac{\varphi'(w)}{\varphi(w) - \varphi(z)} - \frac{1}{w - z}\right] \frac{\bar{\zeta} - \bar{w}}{(1 - \bar{w}\zeta)^2}\, \mathrm{d}A(w)$$

$$= \log \frac{z(\varphi(z) - \varphi(\zeta))}{(z - \zeta)\varphi(z)} - \zeta(1 - |\zeta|^2)\left[\frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z}\right].$$

Next, we compute that

$$\zeta^2 \int_{\mathbb{D}} \frac{\bar{\zeta} - \bar{w}}{(w - z)(1 - \bar{w}\zeta)^2}\, \mathrm{d}A(w) = \log\left(1 - \bar{z}\zeta\right) + \bar{z}\zeta.$$

The assertion is now immediate.   □

We show that the additional term which appears in identity of Proposition 3.2 is uniformly bounded with respect to z and ζ.

Lemma 3.3. *There is an absolute constant $C > 0$ such that*

$$(3.5) \qquad (1 - |\zeta|^2)\left|\frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z}\right| \le C$$

for any $z, \zeta \in \mathbb{D}$.

Proof. By a classical property of the class $\mathcal{S}$, for any $\psi \in \mathcal{S}$, we have

$$\left|w\frac{\psi'(w)}{\psi(w)}\right| \le \frac{4}{1 - |w|^2}, \qquad w \in \mathbb{D},$$

and, consequently,

$$(3.6) \qquad \left|\frac{\psi'(w)}{\psi(w)} - \frac{1}{w}\right| \le \frac{C_1}{1 - |w|^2}, \qquad w \in \mathbb{D},$$

for an absolute constant C_1. Applying (3.6) to

$$\psi(w) = \frac{\varphi\left(\frac{w + z}{1 + \bar{z}w}\right) - \varphi(z)}{(1 - |z|^2)\varphi'(z)}$$

and $\zeta = \frac{w + z}{1 + \bar{z}w}$, we get, for any $z, \zeta \in \mathbb{D}$,

$$\left|\frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z}\frac{1 - |z|^2}{1 - \zeta\bar{z}}\right| \le \frac{C_1}{1 - |\zeta|^2},$$

which, obviously, is equivalent to (3.5). □

4. Marcinkiewicz-Zygmund integral and a Sobolev-type inequality

Consider one more integral transform which appeared in Proposition 3.2. Put

$$\widetilde{\mathfrak{C}}_\varphi[f](z) = \int_\mathbb{D} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\bar{z} - \bar{w}}{1 - \bar{w}z} f(w) \, dA(w).$$

For positive κ we consider the space

$$\mathcal{L}_\kappa(\mathbb{D}) = L^{(2+\kappa)/(1+\kappa)} \Big(\mathbb{D}, (1 - |z|^2)^{-\kappa/(1+\kappa)} \, dA(z) \Big)$$

where the norm is given by

$$\|f\|_{\mathcal{L}_\kappa(\mathbb{D})} = \left(\int_\mathbb{D} |f(z)|^{(2+\kappa)/(1+\kappa)} (1 - |z|^2)^{-\kappa/(1+\kappa)} \, dA(z) \right)^{(1+\kappa)/(2+\kappa)}.$$

Suppose that $f \in \mathcal{L}_\kappa(\mathbb{D})$. Then, by the Hölder inequality,

$$(4.1) \qquad |\widetilde{\mathfrak{C}}_\varphi[f](z)| \leq \left\{ \int_\mathbb{D} \left| \frac{(w - z)\varphi'(w)}{(1 - \bar{w}z)(\varphi(w) - \varphi(z))} \right|^{2+\kappa} (1 - |w|^2)^\kappa \, dA(w) \right\}^{1/(2+\kappa)} \times \|f\|_{\mathcal{L}_\kappa(\mathbb{D})}.$$

The function

$$J_\kappa[\varphi](z) = \int_\mathbb{D} \left| \frac{(w - z)\varphi'(w)}{(1 - \bar{w}z)(\varphi(w) - \varphi(z))} \right|^{2+\kappa} (1 - |w|^2)^\kappa \, dA(w)$$

is essentially the familiar Marcinkiewicz-Zygmund integral. Indeed, if $\Omega = \varphi(\mathbb{D})$, then the Marcinkiewicz-Zygmund integral for Ω is defined by the formula

$$I_\kappa(z) = \int_\Omega \frac{\delta(w)^\kappa}{|z - w|^{2+\kappa} + \delta(w)^{2+\kappa}} \, dA(w),$$

where $\delta(z)$ denotes the Euclidean distance from $z \in \Omega$ to the boundary $\partial\Omega$. It is well known that

$$\frac{1}{4}(1 - |z|^2)|\varphi'(z)| \leq \delta(\varphi(z)) \leq (1 - |z|^2)|\varphi'(z)|, \qquad z \in \mathbb{D}.$$

Changing the variables, it is now easy to check that

$$J_k[\varphi](z) \leq 4^\kappa (1 + \epsilon) I_\kappa(\varphi(z)) + O_{\epsilon,\kappa}(1), \qquad z \in \mathbb{D},$$

holds for each positive ϵ. From the work of Zygmund [12], we know that for bounded Ω,

$$\int_\Omega \exp[c \, I_\kappa(z)] \, dA(z) < +\infty$$

holds for all real $c < c_0$, where c_0 is some positive number that depends on κ. More precisely, one can take $c_0 = \gamma_0 \kappa$, for some positive absolute constant γ_0. This leads to an integral estimate of $J_\kappa[\varphi]$ on $\mathbb{D}$:

$$(4.2) \qquad \int_\mathbb{D} \left| \exp\left[\gamma \, \kappa \, J_\kappa[\varphi](z) \right] \right| |\varphi'(z)|^2 \, dA(z) < +\infty,$$

for $\gamma < \gamma_1 = 4^{-\kappa}(1 + \epsilon)^{-1}\gamma_0$.

Denote by $\mathbf{B}_\kappa$ the unit ball in the space $\mathcal{L}_\kappa(\mathbb{D})$. Now it follows from (4.1) that

$$\sup_{f \in \mathbf{B}_\kappa} \left| \widetilde{\mathfrak{C}}_\varphi[f](z) \right|^{2+\kappa} \leq C J_k[\varphi](z) + O(1), \qquad z \in \mathbb{D}.$$

In view of (4.2) we have proved the following inequality:

Lemma 4.1. *For any κ with $0 < \kappa < 1$, we have for $0 < \gamma < \gamma_1 = 4^{-\kappa}(1 + \epsilon)^{-1}\gamma_0$,*

$$(4.3) \qquad \int_{\mathbb{D}} \exp\left\{\gamma\kappa \sup_{f \in \mathbf{B}_\kappa} \left|\widetilde{\mathfrak{C}}_\varphi[f](z)\right|^{2+\kappa}\right\} |\varphi'(z)|^2 \, dA(z) < +\infty.$$

Remark 4.2. The inequality (4.3) should be compared with the following corollary of the classical Sobolev inequality. For $\zeta \in \mathbb{D}$, consider the function $f_\zeta(z) = \zeta/(1 - \bar{z}\zeta)$. Note that the right-hand side of our basic identity (3.1) is $\mathfrak{C}_\varphi[f_\zeta](z)$. It is easy to see that

$$\|f_\zeta\|_{L^2(\mathbb{D})}^2 = \log\frac{1}{1 - |\zeta|^2}.$$

Now put $\tilde{f}_\zeta = f_\zeta/\|f_\zeta\|_{L^2(\mathbb{D})}$. Then, changing variables in Proposition 2.1, we obtain

$$(4.4) \quad \int_{\mathbb{D}} \exp\left\{\beta_0 \left|\mathfrak{C}_\varphi[\tilde{f}_\zeta](z)\right|^2\right\} |\varphi'(z)|^2 \, dA(z) = \int_{\mathbb{D}} \exp\left\{\beta_0 \frac{|\mathfrak{C}_\varphi[f_\zeta](z)|^2}{\log\frac{1}{1-|\zeta|^2}}\right\} |\varphi'(z)|^2 \, dA(z) < +\infty.$$

As we will see below, if we could establish a "diagonal" analog of inequality (4.4) with $\zeta = z$, this would imply the strong Jones-Makarov estimate (with ε^2 in the error term).

5. Proof of Theorem 1.1.

An application of inequality (4.3). We now specialize to $\zeta = z$ in Proposition 3.2. In view of Lemma 3.3, we obtain

$$(5.1) \qquad \log\frac{z\varphi'(z)}{\varphi(z)} + \log\left(1 - |z|^2\right) + O(1) = z^2 \int_{\mathbb{D}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\bar{z} - \bar{w}}{(1 - \bar{w}z)^2} \, dA(w), \qquad z \in \mathbb{D}.$$

Note that the right-hand side of (5.1) is $\widetilde{\mathfrak{C}}_\varphi[g_z](z)$, with $g_z(w) = z^2(1 - \bar{w}z)^{-1}$. Thus, we have

$$(5.2) \qquad \left|\widetilde{\mathfrak{C}}_\varphi[g_z](z)\right| \leq \left|\log\left(z\frac{\varphi'(z)}{\varphi(z)}(1 - |z|^2)\right)\right| + O(1), \qquad z \in \mathbb{D}.$$

We plan to apply (4.3) to $f = g_z/\|g_z\|_{\mathcal{L}_\kappa(\mathbb{D})}$. First we estimate the norm

$$\|g_z\|_{\mathcal{L}_\kappa(\mathbb{D})}^{2+\kappa} = |z|^{2(1+\kappa)} \left(\int_{\mathbb{D}} \frac{(1 - |w|^2)^{-\kappa/(1+\kappa)}}{|1 - \bar{w}z|^{(2+\kappa)/(1+\kappa)}} \, dA(w)\right)^{1+\kappa}$$

for $1/2 \leq |z| < 1$. Note that for fixed θ with $-\frac{1}{2} < \theta < +\infty$,

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^{2\theta}}{|1 - \bar{w}z|^{2+2\theta}} \, dA(w) = (1 + o(1))\frac{\Gamma(1 + 2\theta)}{\Gamma(1 + \theta)^2} \log\frac{1}{1 - |z|^2}, \qquad |z| \to 1^-.$$

We conclude that there exists an absolute constant $C > 0$ such that

$$(5.3) \qquad \|g_z\|_{\mathcal{L}_\kappa(\mathbb{D})}^{2+\kappa} \geq C\left(\log\frac{1}{1 - |z|^2}\right)^{1+\kappa},$$

for any $1/2 \leq |z| < 1$ and any $\kappa \in (0,1)$ (in what follows we are interested only in small κ). Combining (5.2) and (5.3), we obtain

$$\left| \log \left(z \frac{\varphi'(z)}{\varphi(z)} (1 - |z|^2) \right) \right|^{2+\kappa} \Bigg/ \left(\log \frac{1}{1 - |z|^2} \right)^{1+\kappa}$$

$$\leq C_1 \left| \widetilde{\mathfrak{C}}_\varphi \left[\frac{g_z}{\|g_z\|_{\mathcal{L}_\kappa(\mathbb{D})}} \right] \right|^{2+\kappa} + O(1), \qquad 1/2 \leq |z| < 1,$$

where C_1 is an absolute positive constant. Finally, it follows from (4.3) that

$$(5.4) \quad \int_{\mathbb{D}} \exp \left\{ \gamma \kappa \left| \log \left(z \frac{\varphi'(z)}{\varphi(z)} (1 - |z|^2) \right) \right|^{2+\kappa} \Bigg/ \left(\log \frac{1}{1 - |z|^2} \right)^{1+\kappa} \right\} \mathrm{d}A(z)$$

$$= \int_{\mathbb{D}} \exp \left\{ \gamma \kappa \left| 1 - \frac{\log \frac{z \, \varphi'(z)}{\varphi(z)}}{\log \frac{1}{1 - |z|^2}} \right|^{2+\kappa} \log \frac{1}{1 - |z|^2} \right\} \mathrm{d}A(z) < +\infty.$$

Linear approximation argument. We use a very simple argument to complete the proof of Theorem 1.1. Applying the convexity estimate

$$|a|^{2+\kappa} \geq |b|^{2+\kappa} - (2 + \kappa)|b|^\kappa \operatorname{Re} \left[b(\bar{b} - a) \right]$$
$$= |b|^{2+\kappa} + (2 + \kappa)|b|^\kappa \left[\operatorname{Re} b - |b|^2 \right] - (2 + \kappa)|b|^\kappa \operatorname{Re} \left[b(1 - a) \right], \qquad a, b \in \mathbb{C},$$

to

$$a = 1 - \log \left(z \frac{\varphi'(z)}{\varphi(z)} (1 - |z|^2) \right) \Bigg/ \log \frac{1}{1 - |z|^2},$$

we obtain

$$(5.5) \quad \left| 1 - \log \left(z \frac{\varphi'(z)}{\varphi(z)} (1 - |z|^2) \right) \Bigg/ \log \frac{1}{1 - |z|^2} \right|^{2+\kappa}$$

$$\geq \left[|b|^{2+\kappa} + (2 + \kappa)|b|^\kappa \left[\operatorname{Re} b - |b|^2 \right] \right] \log \frac{1}{1 - |z|^2} - (2 + \kappa)|b|^\kappa \operatorname{Re} \left[b \log \frac{z \varphi'(z)}{\varphi(z)} \right]$$

for any $b \in \mathbb{C}$. When we insert estimate (5.5) into (5.4), we get

$$\int_{\mathbb{D}} \exp \left\{ \gamma \kappa \left[|b|^{2+\kappa} + (2 + \kappa)|b|^\kappa \left[\operatorname{Re} b - |b|^2 \right] \right] \log \frac{1}{1 - |z|^2} \right.$$

$$\left. - \operatorname{Re} \left[\gamma \kappa (2 + \kappa) |b|^\kappa b \log \frac{z \varphi'(z)}{\varphi(z)} \right] \right\} \mathrm{d}A(z) < +\infty.$$

Now we put $\tau = \gamma \kappa (2 + \kappa) |b|^\kappa b$ and choose $\kappa = \left(\log \frac{1}{|b|} \right)^{-1}$ (so that $|b|^\kappa = e^{-1}$). Note also that

$$\exp \left\{ - \operatorname{Re} \left(b(2 + \kappa)|b|^\kappa \log \frac{z \varphi'(z)}{\varphi(z)} \right) \right\} = \left| \left(\frac{z \varphi'(z)}{\varphi(z)} \right)^{-\tau} \right|.$$

Thus, we have

$$(5.6) \quad \int_{\mathbb{D}} \left| \left(\frac{z \varphi'(z)}{\varphi(z)} \right)^{-\tau} \right| (1 - |z|^2)^{- \operatorname{Re} \tau + (1+\kappa)e^{-1}|b|^2} \, \mathrm{d}A(z) < +\infty.$$

To complete the proof, notice that $(1 + \kappa)e^{-1}|b|^2 \asymp |\tau|^2 \log \frac{1}{\tau}$, for our choice of κ, and also that for $\varphi \in \mathcal{S}_b$,

$$(5.7) \qquad\qquad 1/4 \leq |\varphi(z)/z| \leq \|\varphi\|_\infty, \qquad z \in \mathbb{D}.$$

Hence, (5.6) implies (1.3).

Proof of Corollary 1.2. To prove Corollary 1.2, we need to get rid of the factor $z/\varphi(z)$ in (1.3), that is, we need to show that

$$\int_\mathbb{D} \left| (\varphi'(z))^{2-\tau} \right| (1 - |z|^2)^{-\operatorname{Re}\tau + C|\tau|^2 \log \frac{1}{|\tau|}} \, dA(z) < +\infty.$$

Applying the identity in Proposition 3.2 to $z = 0$ and $\zeta = z$, we obtain (making use of Lemma 3.3)

$$\log \frac{\varphi(z)}{z} = z^2 \int_\mathbb{D} \frac{\varphi'(w)}{\varphi(w)} \frac{\bar{z} - \bar{w}}{(1 - \bar{w}z)^2} \, dA(w) + O(1), \qquad z \in \mathbb{D}.$$

As we combine this with (5.1), we get

$$\log \left[\varphi'(z)(1 - |z|^2) \right] = z^2 \int_\mathbb{D} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \left[\frac{2\varphi(w) - \varphi(z)}{\varphi(w)} \right] \frac{\bar{z} - \bar{w}}{(1 - \bar{w}z)^2} \, dA(w) + O(1), \qquad z \in \mathbb{D}.$$

By (5.7), for $|w| \geq 1/2$,

$$\left| \frac{2\varphi(w) - \varphi(z)}{\varphi(w)} \right| \leq 2 + 8\|\varphi\|_\infty, \qquad z \in \mathbb{D}.$$

Also, it is easy to see that

$$\left| z^2 \int_{\{|w|<1/2\}} \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \left[\frac{2\varphi(w) - \varphi(z)}{\varphi(w)} \right] \frac{\bar{z} - \bar{w}}{(1 - \bar{w}z)^2} \, dA(w) \right|$$

$$\leq C_1 \int_{\{|w|<1/2\}} \left| \frac{z - w}{\varphi(z) - \varphi(w)} \right| \frac{dA(w)}{|\varphi(w)|} \leq C_2$$

for any $z \in \mathbb{D}$. We conclude that

$$\left| \log \left[\varphi'(z)(1 - |z|^2) \right] \right| \leq C_3 |z|^2 \int_\mathbb{D} \left| \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\bar{z} - \bar{w}}{(1 - \bar{w}z)^2} \right| dA(w) + O(1), \qquad z \in \mathbb{D}.$$

The rest of the proof is completely analogous to the proofs of Lemma 4.1 and Theorem 1.1 once we notice that (4.1) applies to

$$\widehat{\mathfrak{C}}_\varphi[f](z) = \int_\mathbb{D} \left| \frac{\varphi'(w)}{\varphi(w) - \varphi(z)} \frac{\bar{z} - \bar{w}}{1 - \bar{w}z} \right| |f(w)| \, dA(w)$$

in place of $\widetilde{\mathfrak{C}}_\varphi[f](z)$.

Remark 5.1. If we could establish a "diagonal" analog of inequality (4.4) with $\zeta = z$, this would imply the strong Jones-Makarov estimate. Indeed, by (3.1),

$$\log |\varphi'(z)|(1 - |z|^2) \leq |F_z(z)| + O(1), \qquad z \in \mathbb{D},$$

and, by the linear approximation argument as above, we conclude that

$$\int_\mathbb{D} |\varphi'(z)|^{2-\varepsilon} (1 - |z|^2)^{-\varepsilon + C\varepsilon^2} \, dA(z) < +\infty.$$

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